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Local Spectral Radii and Collatz-Wielandt Numbers of Monic Operator Polynomials With Nonnegative Coefficients

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ABSTRACT

Operator polynomials $L(\lambda) = \lambda^l I - \lambda^{l-1}A_{l-1} - \cdots - \lambda A_1 - A_0$ are considered, where $A_0, \dots, A_{l-1}$ are nonnegative operators in a Banach space $\mathscr{X}$ with normal cone $\mathscr{X}_+$. For $x \in \mathscr{X}_+$ we define the local spectral radius $r_L(x)$ and the lower and upper Collatz-Wielandt numbers $\underline{r}_L(x)$ and $\tilde{r}_L(x)$, respectively, of x with respect to L. We characterize these quantities with the help of corresponding quantities with respect to the first companion operator belonging to L and the operator function $S(\lambda) = A_{l-1} + \lambda^{-1}A_{l-2} + \cdots + \lambda^{-l+1}A_0$. Many properties known in the linear case $l = 1$ have generalizations to the case $l > 1$; e.g., $\underline{r}_L(x) \leqslant r_L(x) \leqslant \tilde{r}_L(x)$ is true for all $x \in \mathscr{X}_+$. From these local results we obtain results for the global spectral radius $r(L)$, which were proved earlier under more restrictive conditions. © 1998 Elsevier Science Inc.

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1. INTRODUCTION

Many spectral properties of a monic operator polynomial

$$L(\lambda) = \lambda^l I - \lambda^{l-1}A_{l-1} - \cdots - \lambda A_1 - A_0, \qquad \lambda \in \mathbb{C}, \tag{1}$$

where $A_0, \dots, A_{l-1}$ are linear bounded operators acting in a Banach space $\mathscr{X}$, can be analyzed with the help of the first companion operator

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I \\ A_0 & A_1 & \cdots & A_{l-2} & A_{l-1} \end{bmatrix} : \mathscr{X}^l \to \mathscr{X}^l. \tag{2}$$

It is known, e.g., that the spectra of L and C coincide. If $\mathscr{X}$ is an ordered Banach space with normal closed cone $\mathscr{X}_+$ and the operators A_j are nonnegative with respect to $\mathscr{X}_+$, i.e., $A_j \mathscr{X}_+ \subset \mathscr{X}_+$, $j = 0, \dots, l-1$, then not only does the linearization by the companion operator C play a role, but also the spectral properties of the operator

$$S(\lambda) = A_{l-1} + \frac{1}{\lambda} A_{l-1} + \cdots + \frac{1}{\lambda^{l-2}} A_1 + \frac{1}{\lambda^{l-1}} A_0, \qquad \lambda \neq 0; \tag{3}$$

see [6, 8, 14, 18]. In particular the peripherical spectra of L and of $S(\rho)$, where ρ is the spectral radius of L, have many common properties. This can be considered as another form of linearization of the polynomial L.

In the spectral theory of nonnegative operators the local spectral theory also is of some interest; see [4, 5, 16]. It is the goal of this paper to study the local spectral radius and its relation to the Collatz-Wielandt numbers of $x \in \mathscr{X}_+$ with respect to L, i.e., we consider the following three quantities:

$$r_L(x) = \inf\{\rho > 0 : L^{-1}(\cdot)x \text{ has a holomorphic extension to } \times\{\lambda \in \mathbb{C} : |\lambda| > \rho\}\},$$

$$\underline{r}_L(x) = \sup\{\lambda \geqslant 0 : -L(\lambda)x \in \mathscr{X}_+\},$$

$$\tilde{r}_L(x) = \inf\{\lambda \geqslant 0 : L(\lambda)x \in \mathscr{X}_+\}.$$

If $\mathscr{X}_+$ is a normal cone in $\mathscr{X}$ and the coefficients A_j $(j = 0, \dots, l-1)$ are nonnegative, then, as we shall show,

$$\underset{\sim}{r}_L(x) \leqslant r_L(x) \leqslant \tilde{r}_L(x) \qquad \text{for all} \quad x \in \mathscr{X}_+, \quad x \neq 0.$$

This inequality is well known for the "linear" polynomial $L(\lambda) = \lambda I - A$, (see [5]) and gives under certain conditions on x or A the estimates for the spectral radius for A of L. Collatz and H. Wielandt. It has been used in [10] and [14] to estimate the spectral radius of L.

In Section 2 we prove some connections between the local spectral radius $r_L(x)$ and its linearization with the companion operator C.

In Section 3 we characterize $r_L(x)$ for $x \in \mathscr{X}_+$ with the help of the local spectral radius of x with respect to the operator $S(\lambda)$. We obtain local versions of results known for the global spectral radius.

In Section 4 we again characterize the lower and upper Collatz-Wielandt numbers $\underset{\sim}{r}_L(x)$ and $\tilde{r}_L(x)$ of x with respect to L by corresponding quantities with respect to C and $S(\lambda)$, respectively. We prove the fundamental relation between these Collatz-Wielandt numbers and the local spectral radius mentioned above. At the end of the paper we indicate how we can generalize the iterative methods known for linear operator polynomials to those of arbitrary degree.

For the spectral theory of operator polynomials we use the terminology of [19]; for the theory of ordered Banach spaces and for the spectral theory of (cone-)nonnegative operators in such spaces we use the terminology of [20].

2. LOCAL SPECTRAL RADII WITH RESPECT TO OPERATOR POLYNOMIALS

Let L be a monic operator polynomial of degree l as in (1). Its spectrum is the set

$$\Sigma(L) = \{\lambda \in \mathbb{C} : L(\lambda) \text{ is not two-sided invertible in } \mathscr{L}(\mathscr{X})\}.$$

Here $\mathscr{L}(\mathscr{X})$ denotes the Banach algebra of all bounded linear operators acting in $\mathscr{X}$. This spectrum $\Sigma(L)$ coincides with the spectrum $\sigma(C)$ of the companion operator C given in (2); therefore $\Sigma(L)$ is a compact subset of $\mathbb{C}$. The map

$$L^{-1} : \mathbb{C} \setminus \Sigma(L) \qquad \text{with} \quad \lambda \to L^{-1}(\lambda) = \text{the inverse of } L(\lambda)$$

is analytic and vanishes at infinity in order l. The Taylor expansion of L^{-1} at infinity is given by

$$L^{-1}(\lambda) = \lambda^{-l} \sum_{k=0}^{\infty} \lambda^{-k} L_k \qquad \text{if} \quad |\lambda| > r(L), \tag{4}$$

where $r(L) = \max\{|\lambda| : \lambda \in \Sigma(L)\}$ is the spectral radius of L, and $L_k \in \mathscr{L}(\mathscr{X})$, $k = 0, 1, \dots$. The series in (4) converges in $\mathscr{L}(\mathscr{X})$ with respect to the operator norm.

Properties of the functions $L^{-1}(\cdot)x : \mathbb{C} \setminus \Sigma(L) \to \mathscr{X}$ for $x \in \mathscr{X}$ play a fundamental role in local spectral theories. For the case that L is linear in λ [i.e. $L(\lambda) = \lambda I - A$] see [1], [2], and [13] for example. We will not develop a general local spectral theory for operator polynomials in this paper. We will deal here only with the concept of the local spectral radius $r_L(x)$ of a vector $x \in \mathscr{X}$ with respect to the operator polynomial L, which is defined in the following way:

$$r_L(x) = \inf\{r > 0 :$$
$$L^{-1}(\cdot)x \text{ has a holomorphic extension to } \{\lambda \in \mathbb{C} : r < |\lambda|\}\}. \tag{5}$$

With (4) we obtain $r_L(x) = \limsup_{k\to\infty} \|L_k x\|^{1/k}$. The function

$$x_L : \{\lambda \in \mathbb{C} : r_L(x) < |\lambda|\} \to \mathscr{X} \qquad \text{with} \quad x_L(\lambda) = \lambda^{-l} \sum_{k=0}^{\infty} \lambda^{-k} L_k x \tag{6}$$

is the unique holomorphic extension of $L^{-1}(\cdot)x$ defined on the complement of the closed disc with radius $r_L(x)$ and center zero.

PROPOSITION 2.1. *Let L be a monic operator polynomial. Then:*

(I) *For all $x \in \mathscr{X}$ we have $L(\lambda)x_L(\lambda) = x$ if $r_L(x) < |\lambda|$.*

(II) *For all $x \in \mathscr{X}$, $x \neq 0$, there exists at least one point in $\Sigma(L)$ with modulus $r_L(x)$.*

(III) $r(L) = \max\{r_L(x) : x \in \mathscr{X}\}$.

Proof. (I) follows because x_L is a holomorphic extension of $L^{-1}(\cdot)x$.

To prove (II) assume that the circle $\Gamma = \{\lambda : |\lambda| = r_L(x)\}$ is disjoint from $\Sigma(L)$; then $L^{-1}(\cdot)x$ is defined and holomorphic in a neighborhood of Γ. Since x_L is a holomorphic extension of $L^{-1}(\cdot)x$ to $\{\lambda \in \mathbb{C} : r_L(x) < |\lambda|\}$, this latter function has a holomorphic extension to $\{\lambda \in \mathbb{C} : r < |\lambda|\}$ with $r < r_L(x)$. This contradicts the definition of $r_L(x)$; therefore the assumption cannot hold.

(III): Of course, $r_L(x) \leqslant r(L)$ for all $x \in \mathscr{X}$. Assume $r_L(x) < r(L)$ for all $x \in \mathscr{X}$. Then for all λ with $|\lambda| = r(L)$ the operators $L(\lambda)$ are surjective by (I). Therefore

$$\{\lambda \in \mathbb{C} : r(L) \leqslant |\lambda|\} \subseteq \{\lambda \in \mathbb{C} : L(\lambda) \text{ is a semi-Fredholm operator}\}.$$

On connected components of the last set, index $L(\lambda) = \dim\ker L(\lambda) - \operatorname{codim}\operatorname{ran} L(\lambda)$ is constant. For $r(L) < |\lambda|$ we have index $L(\lambda) = \dim\ker L(\lambda) = \operatorname{codim}\operatorname{ran} L(\lambda) = 0$; for $r(L) = |\lambda|$ we have index $L(\lambda) = 0$ and $\operatorname{codim}\operatorname{ran} L(\lambda) = 0$; thus $\dim\ker L(\lambda) = 0$. This means $L(\lambda)$ is invertible for all λ with $|\lambda| = r(L)$. We obtain a contradiction to the definition of $r(L)$; therefore the assumption cannot hold. ∎

In the linear case $L(\lambda) = \lambda I - A$ we write (as usual) $\sigma(A)$, $r(A)$, $R(\cdot, A)$, $r_A(x)$, and x_A instead of $\Sigma(L)$, $r(L)$, $L^{-1}(\cdot)$, $r_L(x)$, and x_L, respectively.

We will now characterize the local results from above with the help of the companion operator C defined in (2). For this purpose it is convenient to represent C as a 2×2 operator matrix

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \quad \text{where} \quad C_{11} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & & I & & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & I \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad C_{12} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ I \end{bmatrix},$$

$C_{21} = [A_0, A_1, \ldots, A_{l-2}]$, and $C_{22} = A_{l-1}$. For $\lambda \notin \sigma(C) = \Sigma(L)$ and $\lambda \neq 0$ we have (see [18, p. 480])

$$\lambda I - S(\lambda) \text{ is invertible, where } S \text{ is defined in } (3),$$

$$R(\lambda, C) = \begin{bmatrix} R(\lambda, C_{11})[I + C_{12}R(\lambda, S(\lambda))C_{21}R(\lambda, C_{11})] & R(\lambda, C_{11})C_{12}R(\lambda, S(\lambda)) \\ R(\lambda, S(\lambda))C_{21}R(\lambda, C_{11}) & R(\lambda, S(\lambda)) \end{bmatrix}, \tag{7}$$

$$R(\lambda, S(\lambda)) = [\lambda I - S(\lambda)]^{-1} = \lambda^{l-1}L^{-1}(\lambda).$$

The assertions of the next proposition follow immediately from Proposition 2.1 and (7).

PROPOSITION 2.2. *Let L be a monic operator polynomial and C its companion operator. Then*

(I) $r_L(x) = r_C(\mathrm{col}[0,\dots,0,x]) = \limsup_{k\to\infty} \|C^k \mathrm{col}[0,\dots,0,x]\|^{1/k}$,

(II) $\mathrm{col}[0,\dots,0,x]_C(\lambda) = \mathrm{col}[x_L(\lambda), \lambda^{-1}x_L(\lambda),\dots,\lambda^{l-1}x_L(\lambda)]$ *if* $r_L(x) < |\lambda|$,

(III) $r(C) = \max\{r_C(\mathrm{col}[0,\dots,0,x]) : x \in \mathscr{X}\}$.

3. LOCAL SPECTRAL RADII FOR NONNEGATIVE VECTORS WITH RESPECT TO OPERATOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS

In this section our goal is to describe relations between the local spectral radii $r_L(x)$ and $r_{S(\lambda)}(x)$, if x is an element in a normal closed cone $\mathscr{X}_+$ of $\mathscr{X}$ and the coefficients A_j are nonnegative, i.e. $A_j\mathscr{X}_+ \subset \mathscr{X}_+$ $(j = 0\ 1,\dots,l-1)$. This can be considered as another type of linearization of the operator polynomial L. For the spectral radius of L this has been done in [6], [14], [18]. We obtain some of these global results via Proposition 2.1(III) from the local results under more general conditions.

THEOREM 3.1. *Let $\mathscr{X}$ be an ordered Banach space with normal cone $\mathscr{X}_+$; let L be the monic operator polynomial given by* (1) *with nonnegative coefficients A_j $(j = 0,\dots,l-1)$ and $x \in \mathscr{X}_+$, $x \neq 0$. Then we have:*

(I) *The function $]0,\infty[\to \mathbb{R}_+$ with $\lambda \mapsto r_{S(\lambda)}(x)$ is continuous and nonincreasing.*

(II) *If $r_L(x) > 0$ and $\rho > 0$, then*

$$r_L(x) = \rho \quad \textit{if and only if} \quad r_{S(\rho)}(x) = \rho. \tag{8}$$

(III) $r_L(x) = 0$ *if and only if* $r_{S(\lambda)}(x) = 0$ *for all* $\lambda > 0$.

Proof. If $0 < \lambda \leqslant \mu$ we have (see [18, p. 481])

$$0 \leqslant \left(\frac{\lambda}{\mu}\right)^{l-1} S(\lambda) \leqslant S(\mu) \leqslant S(\lambda) \leqslant \left(\frac{\mu}{\lambda}\right)^{l-1} S(\mu). \tag{9}$$

By the monotonicity of the local spectral radius [note that $0 \leqslant S \leqslant T$ and $0 \leqslant x \leqslant y$ imply $r_S(x) \leqslant r_T(y)$ if $\mathscr{X}_+$ is normal] we obtain

$$0 \leqslant \left(\frac{\lambda}{\mu}\right)^{l-1} r_{S(\lambda)}(x) \leqslant r_{S(\mu)}(x) \leqslant r_{S(\lambda)}(x) \leqslant \left(\frac{\mu}{\lambda}\right)^{l-1} r_{S(\mu)}(x). \quad (10)$$

This inequality gives us both assertions of (I). Parts (II) and (III) are easy consequences of (I) and the following claim:

If η is a positive number, then $r_L(x) < \eta$ and $r_{S(\eta)}(x) < \eta$ are equivalent.

First we show that $r_{S(\lambda)}(x) \leqslant r_{S(|\lambda|)}(x)$ for all $\lambda \neq 0$. Let $\varepsilon > 0$; then there exists a $\gamma = \gamma(\varepsilon) > 0$ such that $\|S^k(|\lambda|)x\| \leqslant \gamma[r_{S(|\lambda|)}(x) + \varepsilon]^k$ for $k = 1, 2, \dots$. Now $A_j \geqslant 0$ $(j = 0, \dots, l-1)$ and $x \geqslant 0$ imply $|\langle S^k(\lambda)x, x'\rangle| \leqslant \langle S^k(|\lambda|)x, x'\rangle$ for all $x' \in \mathscr{X}'_+$. Now $\mathscr{X}'_+$ is generating, since $\mathscr{X}_+$ is normal (see [20, p. 218]); thus there exists a $\mu > 0$ such that for all $x' \in \mathscr{X}'$ and all $k = 1, 2, \dots$

$$|\langle S^k(\lambda)x, x'\rangle| \leqslant |\langle S^k(|\lambda|)x, x'\rangle| \leqslant \mu\gamma\left[r_{S(|\lambda|)}(x) + \varepsilon\right]^k \|x'\|.$$

Therefore $r_{S(\lambda)}(x) = \limsup_{k\to\infty} \|S^k(\lambda)x\|^{1/k} \leqslant r_{S(|\lambda|)}(x) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $r_{S(\lambda)}(x) \leqslant r_{S(|\lambda|)}(x)$.

Assume $r_{S(\eta)}(x) < \eta$. By (I) we can find a μ such that $r_{S(\eta)}(x) \leqslant r_{S(\mu)}(x) < \mu < \eta$. If $\mu \leqslant |\lambda|$, then $r_{S(\lambda)}(x) \leqslant r_{S(|\lambda|)}(x) \leqslant r_{S(\mu)}(x) < |\lambda|$, $\sum_{k=0}^{\infty}(S(\lambda)/\lambda)^n x$ converges uniformly for $|\lambda| \geqslant \mu$, and

$$\lambda^{-1}\sum_{k=0}^{\infty} \lambda^{-k}S^k(\lambda)x = [\lambda I - S(\lambda)]^{-1}x = \lambda^{-l+1}L^{-1}(\lambda)x$$

if $r(L) < |\lambda|$.

Therefore $L^{-1}(\cdot)x$ has a holomorphic extension to $\{\lambda \in \mathbb{C} : |\lambda| > \mu\}$, and we obtain $r_L(x) \leqslant \mu < \eta$.

Assume now $r_L(x) < \eta$. In $\{\lambda \in \mathbb{C} : r_L(x) < |\lambda|\}$ the function x_L is analytic. Since $\lambda^{-1}S(\lambda) \to 0$ for $|\lambda| \to \infty$, we can find a $\rho > r(L)$ such that for all λ with $|\lambda| > \rho$ we have

$$x_L(\lambda) = L^{-1}(\lambda)x = \lambda^{-l+1}[\lambda I - S(\lambda)]^{-1}x = \lambda^{-l}\sum_{k=0}^{\infty} \lambda^{-k}S^k(\lambda)x.$$

By the next proposition (see Remark 3.3), for all λ with $r_L(x) < |\lambda|$ the last series converges and the last equality is true. Therefore $\lambda^{-k}S^k(\lambda)x \to 0$ if $k \to \infty$ and $r_L(x) < |\lambda|$. Thus we have $r_{S(\lambda)}(x) \leqslant |\lambda|$. Now choose a λ such that $r_L(x) < \lambda < \eta$. Using (I), we obtain $r_{S(\eta)}(x) \leqslant r_{S(\lambda)}(x) \leqslant \lambda < \eta$. The claim is proved. ■

For the proof of the following proposition we thank G. Frank and J. R. Langley.

PROPOSITION 3.2. *Let f be a complex valued function holomorphic in $\{z \in \mathbb{C} : |z| < r\}$. Let $\rho \leqslant r$ and*

$$f(z) = \sum_{k=0}^{\infty} p_k(z) \qquad \textit{for all} \quad z \in \mathbb{C} \quad \textit{with } |z| < \rho,$$

where each p_k is a polynomial with nonnegative coefficients, and the series converges uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < \rho\}$. Then the series converges uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < r\}$ to $f(z)$.

Proof. It is sufficient to show that the series converges uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < r\}$. Then we can use the permanence property of functional equations under analytic extension to get that its limit is $f(z)$. Since the coefficients of the polynomials p_k are nonnegative, we have $|p_k(z)| \leqslant p_k(x)$ for all $z \in \mathbb{C}$ with $|z| \leqslant x$. Therefore it is sufficient to prove that $\sum_{k=0}^{\infty} p_k(x)$ converges for all x with $0 < x < r$.

Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ for $|z| < r$. From $a_j = \sum_{k=0}^{\infty} (1/j!) p_k^{(j)}(0)$ we obtain $a_j \geqslant 0$, $j = 0, 1, \dots,$ using the assumption that all polynomials p_k have nonnegative coefficients. Given $\varepsilon > 0$ and x with $0 < x < r$, choose a natural number n_0 such that $\sum_{j > n_0} a_j x^j < \varepsilon$. Let $p_k = q_k + r_k$, where q_k is the polynomial of degree less than or equal to n_0 with the corresponding coefficients of p_k. We get $\sum_{k=0}^{\infty} r_k(x) = \sum_{j > n_0} a_j x^j$. Choose $\eta > 1$ such that $x/\eta < \rho$; then $q_k(x) \leqslant \eta^{n_0} q_k(x/\eta) \leqslant \eta^{n_0} p_k(x/\eta)$ for $k = 1, 2, \dots$. Now choose n_1 such that $\sum_{k > n_1} p_k(x/\eta) < \eta^{-n_0}\varepsilon$. Finally, set $n_\varepsilon = \max\{n_0, n_1\}$; then for $n_\varepsilon < m < n$ we obtain

$$\sum_{k=m}^{n} p_k(x) \leqslant \eta^{n_0} \sum_{k > n_1} p_k\left(\frac{x}{\eta}\right) + \sum_{k=0}^{\infty} r_k(x) < 2\varepsilon.$$

Therefore the series $\sum_{k=0}^{\infty} p_k(x)$ converges. ■

REMARK 3.3. If $\mathscr{X}$ is an ordered Banach space with normal cone $\mathscr{X}_+$, then Proposition 3.2 is true for functions f and polynomials p_k with coefficients in $\mathscr{X}_+$ (see [20, p. 262]). In the proof of Theorem 3.1 we apply this vector valued version of Proposition 3.2 to $f(1/z)$ if $|z| > \rho$ and $r \leqslant \rho$.

COROLLARY 3.4. *Let* $\mathscr{X}_+$ *normal and* $x > 0$. *Then*:

(I) $r_L(x)$ *is a singularity of* x_L.

(II) $r_L(x) = \inf\{\lambda > 0 : r_{S(\lambda)}(x) \leqslant \lambda\} = \sup\{\lambda > 0 : \lambda \leqslant r_{S(\lambda)}(x)\}$; *if* $r_L(x)$ *is positive, we can replace the infimum and the supremum with the minimum and maximum, respectively.*

If $\lambda > 0$, *then*:

(III) $r_{S(\lambda)}(x) \leqslant \lambda$ *iff* $r_{S(\lambda)}(x) \leqslant r_L(x) \leqslant [r_{\lambda^{l-1}S(\lambda)}(x)]^{1/l} \leqslant \lambda$.

(IV) $r_{S(\lambda)}(x) \geqslant \lambda$ *iff* $r_{S(\lambda)}(x) \geqslant r_L(x) \geqslant [r_{\lambda^{l-1}S(\lambda)}(x)]^{1/l} \geqslant \lambda$.

Proof. (I) follows from Proposition 2.2 and [4, Theorem 6].

(II) is an easy consequence of Theorem 3.1 and the following simple fact: If $f :]0, \infty[\to \mathbb{R}_+$ is a continuous, decreasing function which does not vanish identically, then there exists a unique $\rho > 0$ such that

$$f(\rho) = \rho \text{ and } f(\lambda) < \lambda\, [\text{or} f(\lambda) > \lambda] \text{ iff } \lambda > \rho\, [\text{or } \lambda < \rho, \text{respectively}].$$

(III): Let $r_{S(\lambda)}(x) \leqslant \lambda$. By (II) we have $r_{S(\lambda)}(x) \leqslant r_L(x)$. Furthermore, $r_{\lambda^{l-1}S(\lambda)}(x) = \lambda^{l-1} r_{S(\lambda)}(x) \leqslant \lambda^l$. If $r_{S(\lambda)}(x) = \lambda$, we have equalities at each place in (III). Assume $r_{S(\lambda)}(x) < \lambda$; then $[r_{\lambda^{l-1}S(\lambda)}(x)]^{1/l} < \lambda$. Let μ lie between these numbers. Since the map $]0, \infty[\to \mathscr{X}_+$ with $\tau \mapsto \tau^{l-1}S(\tau)x$ is increasing and $\mathscr{X}_+$ is normal, we obtain $\mu^{l-1} r_{S(\mu)}(x) \leqslant \lambda^{l-1} r_{S(\lambda)}(x) < \mu^l$. Therefore $r_{S(\mu)}(x) \leqslant \mu$, and $r_L(x) \leqslant \mu$, by (II). If μ approaches $[r_{\lambda^{l-1}S(\lambda)}(x)]^{1/l}$ from above, we obtain $r_L(x) \leqslant [r_{\lambda^{l-1}S(\lambda)}(x)]^{1/l}$. The reverse implication in (III) is trivial.

(IV) can be proved similarly. ∎

The following corollary was proved for the spectral radius of matrix polynomials in [3, Lemma 3].

COROLLARY 3.5. *Let L and M be monic operator polynomials of degree l with nonnegative coefficients, i.e.,*

$$L(\lambda) = \lambda^l I - \lambda^{l-1} A_{l-1} - \cdots - A_0,$$

$$M(\lambda) = \lambda^l I - \lambda^{l-1} B_{l-1} - \cdots - B_0,$$

where A_j and B_j are nonnegative operators in $\mathcal{L}(X)$, $j = 0, \dots, l-1$, and $\mathscr{X}$ is an ordered Banach space with normal cone $\mathscr{X}_+$.

Let $x \in \mathscr{X}_+$ satisfy

$$\sum_{j=0}^{k} A_j x \leqslant \sum_{j=0}^{k} B_j x, \qquad k = 1, \dots, l-2.$$

Then

(I) $r_L(x) \leqslant r_M(x)$ if $r_M(x) \leqslant 1$ and $\sum_{j=0}^{l-1} A_j x \leqslant \sum_{j=0}^{l-1} B_j x$, and
(II) $r_L(x) \geqslant r_M(x)$ if $r_M(x) \geqslant 1$ and $\sum_{j=0}^{l-1} A_j x \geqslant \sum_{j=0}^{l-1} B_j x$.

Proof. For $\lambda > 0$ let $S(\lambda)$ be defined as in (3), and

$$T(\lambda) = B_{l-1} + \frac{1}{\lambda} B_{l-2} + \cdots + \frac{1}{\lambda^{l-1}} B_0.$$

Let the assumptions of (I) be fulfilled. If $0 < \lambda \leqslant 1$, then

$$\begin{aligned} S(\lambda)x &= (A_{l-1} + \cdots + A_0)x \\ &\quad + \left(\frac{1}{\lambda} - 1\right)(A_{l-2} + \cdots + A_0)x + \cdots + \left(\frac{1}{\lambda^{l-1}} - \frac{1}{\lambda^{l-2}}\right) A_0 x \\ &\leqslant (B_{l-1} + \cdots + B_0)x + \left(\frac{1}{\lambda} - 1\right)(B_{l-2} + \cdots + B_0)x \\ &\quad + \cdots + \left(\frac{1}{\lambda^{l-1}} - \frac{1}{\lambda^{l-2}}\right) B_0 x \\ &= T(\lambda)x, \end{aligned}$$

and we get $r_{S(\lambda)}(x) \leqslant r_{T(\lambda)}(x)$, since $\mathscr{X}_+$ is normal. If $r_M(x) = 0$, then $r_{T(\lambda)}(x) = 0$ for all $\lambda > 0$, by Theorem 3.1(III). Therefore $r_{S(\lambda)}(x) = 0$ for $0 < \lambda \leqslant 1$; thus $r_L(x) = 0$, again by Theorem 3.1(I) and (III). If $0 < r_M(x) \leqslant 1$, take $\rho = r_M(x)$ and apply Theorem 3.1(II) to obtain $r_{S(\rho)}(x) \leqslant r_{T(\rho)}(x) = \rho$. Then $r_L(x) \leqslant \rho = r_M(x)$, by Corollary 3.4(III).

If the assumptions of (II) are fulfilled, we obtain $S(\lambda)x \geqslant T(\lambda)x$ for all $\lambda \geqslant 1$. Again with Theorem 3.1 and Corollary 3.4, we get $r_L(x) \geqslant r_M(x)$. ∎

From these results on the local spectral radii we can obtain [as in the linear case $L(\lambda) = \lambda I - A$] corresponding results on the global spectral radius of the operator polynomial. Let $\mathscr{X}$ be an ordered Banach space with normal generating cone $\mathscr{X}_+$; then the cone of all nonnegative operators in $\mathscr{L}(\mathscr{X})$ is a normal cone in $\mathscr{L}(\mathscr{X})$ [20, p. 226]. From (9) we get that the function $]0,\infty[\to \mathscr{R}_+$ with $\lambda \mapsto r(S(\lambda))$ is decreasing and continuous. Therefore the results of Theorem 3.1 and its corollaries have analogues for the global spectral radii of L and $S(\lambda)$. Special cases of these global results were proved in [6] and [18].

In Section 4 we want to compare our results on Collatz-Wielandt numbers for operator polynomials having nonnegative coefficients with special cases considered in [10] and [14]. For this purpose we need some conditions on x and $S(\cdot)$ which imply $r_L(x) = r(L)$. For the linear case $L(\lambda) = \lambda I - A$ such conditions can be found in [7]. The following theorem contains generalizations of some of these conditions to operator polynomials with nonnegative coefficients.

THEOREM 3.6. *Assume that $\mathscr{X}$ is an ordered Banach space with normal cone $\mathscr{X}_+$ and L is an operator polynomial as given in* (1) *with nonnegative coefficients $A_j \in \mathscr{L}(\mathscr{X})$, $j = 0, \dots, l-1$, with $\rho = r(L) > 0$. Let one of the following conditions be satisfied*:

(I) *$\rho = r(L)$ is a pole of the resolvent $R(\cdot, S(\rho))$ of $S(\rho)$.*

(II) *$\mathscr{X}_+$ is solid.*

(III) *$S(\rho)$ is u-bounded from above by an element in the generating cone $\mathscr{X}_+$, i.e., there exists a $u \in \mathscr{X}_+$ such that for every nonzero $x \in \mathscr{X}_+$ a natural number m and a positive number β can be found such that $A^m x \leqslant \beta\mu$ (see* [12, *p.* 59]).

Then

(a) *$r_L(x) = r(L)$ for all $x \in \mathscr{X}_+$ which are quasiinterior elements in the sense of* [20, *p.* 24];

(b) *$r_L(x) = r(L)$ for all $x \in \mathscr{X}_+ \setminus \{0\}$, if $S(\rho)$ is irreducible in the sense of* [20, *p.* 269].

Proof. By [7, Propositions 1, 3, 5, Theorem 7] we have $r_{S(\rho)}(x) = r(S(\rho)) = \rho$. From Theorem 3.1 and its global analogues we obtain $r_L(x) = r_{S(\rho)}(x) = \rho = r(L)$. ∎

REMARK 3.7. From (9) it follows immediately that $S(\lambda)$ is irreducible for all $\lambda > 0$ if and only if $S(\lambda)$ is irreducible for one $\lambda > 0$. For example, irreducibility of $S(\lambda)$ does not imply irreducibility of the companion operator C; see [8].

4. COLLATZ-WIELANDT NUMBERS FOR NONNEGATIVE VECTORS WITH RESPECT TO OPERATOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS

In this section we will define the upper and lower Collatz-Wielandt numbers of a vector $x \in \mathscr{X}_+$ with respect to the monic operator polynomial L as in (1) with nonnegative coefficients A_j. Again we characterize these numbers by the corresponding Collatz-Wielandt numbers of $\mathrm{col}[x, \lambda x, \dots, \lambda^{l-1}x]$ with respect to the companion operator C [defined in (2)] and of x with respect to the nonnegative operator $S(\lambda)$ given in (3).

Let $\mathscr{X}$ be an ordered Banach space with normal cone $\mathscr{X}_+$. For a nonnegative operator A in $\mathscr{X}$ (i.e. $A\mathscr{X}_+ \subset \mathscr{X}_+$) and $x \in \mathscr{X}_+$, the lower and the upper Collatz-Wielandt numbers of x with respect to A are defined by

$$\underline{r}_A(x) = \sup\{\lambda \geqslant 0 : \lambda x \leqslant Ax\},$$

$$\bar{r}_A(x) = \inf\{\lambda \geqslant 0 : Ax \leqslant \lambda x\},$$

respectively. Note that here and in what follows $u \leqslant v$ for $u, v \in \mathscr{X}$ means $v - u \in \mathscr{X}_+$. We use the convention that the infimum of the empty set is ∞; thus $0 \leqslant \underline{r}_A(x) \leqslant \infty$ and $0 \leqslant \bar{r}_A(x) \leqslant \infty$. Before we extend this definition of the Collatz-Wielandt numbers and other characterizations of them to operator polynomials, we make some preliminary remarks.

By $\mathscr{X}'$ and $\mathscr{X}'_+$ we denote in the following the dual space of $\mathscr{X}$ and the dual cone, respectively; $\langle\ ,\ \rangle$ denotes the dual pairing on $\mathscr{X} \times \mathscr{X}'$. Let $x \in \mathscr{X}_+$ and $x' \in \mathscr{X}'_+$ be such that $\langle x, x'\rangle > 0$. By induction on the degree l of the operator polynomial L with nonnegative coefficients it follows that the (scalar valued) polynomial

$$\lambda \mapsto \langle L(\lambda)x, x'\rangle = \lambda^l\langle x, x'\rangle - \lambda^{l-1}\langle A_{l-1}x, x'\rangle - \cdots - \langle A_0 x, x'\rangle$$

has at least one nonnegative root and at most one positive root. By $\lambda_{L,x,x'}$ we denote the greatest nonnegative root of the polynomial above. Then $\langle L(\lambda)x, x'\rangle \geqslant 0$ if $\lambda \geqslant \lambda_{L,x,x'}$, and $\langle L(\lambda)x, x'\rangle < 0$ if $0 < \lambda < \lambda_{L,X,x'}$.

PROPOSITION 4.1. *Let $\mathscr{X}$ be an ordered Banach space with normal cone $\mathscr{X}_+$, let L be the monic operator polynomial given in* (1) *with nonnegative coefficients A_j ($j = 0, \dots, l-1$) and let $x \in \mathscr{X}_+$, $x \neq 0$. Then the following* *six (nonnegative) numbers are equal*:

$$\begin{aligned}
&\sup\{\lambda \geqslant 0 : L(\lambda)x \leqslant 0\},\\
&\sup\{\lambda > 0 : \lambda x \leqslant S(\lambda)x\},\\
&\sup\{\lambda > 0 : \lambda \leqslant \underline{r}_{S(\lambda)}(x)\},\\
&\sup\{\min\{\lambda, \underline{r}_{S(\lambda)}(x)\} : \lambda > 0\},\\
&\sup\{\underline{r}_C(\mathrm{col}[x, \lambda x, \dots, \lambda^{l-1}x]) : \lambda \geqslant 0\},\\
&\inf\{\lambda_{L,x,x'} : x' \in \mathscr{X}'_+, \langle x, x'\rangle > 0\},
\end{aligned}$$

here we use the convention $\sup\{\lambda > 0 : \cdots\} = 0$ *if* $\{\lambda > 0 : \cdots\}$ *is empty*.

Proof. Denote by $r_1, \dots, r_6$ the numbers defined above in that order. The proofs of the equality $r_1 = r_2 = r_3$ and the inequality $r_3 \leqslant r_4$ are trivial. It is not difficult to see that

$$\min\{\lambda, \underline{r}_{S(\lambda)}(x)\} = \underline{r}_C(\mathrm{col}[x, \lambda x, \dots, \lambda^{l-1}x] \qquad \text{for} \quad \lambda > 0;$$

thus $r_4 = r_5$ follows immediately. Next we prove $r_4 \leqslant r_1$. We can and do assume that $0 < r_4$. Let $\lambda > 0$ be such that $\rho = \min\{\lambda, \underline{r}_{S(\lambda)}(x)\} > 0$, then $\rho x \leqslant \underline{r}_{S(\lambda)}(x) \leqslant S(\lambda)x \leqslant S(\rho)x$. Therefore $L(\rho)x \leqslant 0$, i.e. $\rho \leqslant r_1$. For $\lambda \geqslant 0$ and $x' \in \mathscr{X}'_+$ with $L(\lambda)x \leqslant 0$ and $\langle x, x'\rangle > 0$ it follows that $\langle L(\lambda)x, x'\rangle \leqslant 0$; therefore $\lambda \leqslant \lambda_{L,x,x'}$. This proves $r_1 \leqslant r_6$. To prove $r_6 \leqslant r_1$, let $x' \in \mathscr{X}'_+$ be such that $\langle x, x'\rangle > 0$. Then $\langle L(\lambda)x, x'\rangle \leqslant 0$ if $0 \leqslant \lambda \leqslant \lambda_{L,x,x'}$. But if $x' \in \mathscr{X}'_+$ with $\langle x, x'\rangle = 0$, then $\langle L(\lambda)x, x'\rangle = -\langle \sum_{j=0}^{l-1} \lambda^j A_j x, x'\rangle \leqslant 0$ for all $\lambda > 0$. Therefore $\langle L(\lambda)x, x'\rangle \leqslant 0$ for all λ with $0 \geqslant \lambda \geqslant r_6$ and all $x' \in \mathscr{X}'_+$, which is equivalent to $L(\lambda)x \leqslant 0$ for all such λ. Therefore $r_6 \leqslant r_1$. ■

If $x \in \mathscr{X}_+$ and $x' \in \mathscr{X}'_+$ with $\langle x, x'\rangle = 0$, we obtain $\langle L(\lambda)x, x'\rangle \leqslant 0$ for all $\lambda \geqslant 0$. In this case we define

$$\lambda_{L,x,x'} = \begin{cases} 0 & \text{if } \langle L(\lambda)x, x'\rangle = 0 \text{ for all } \lambda \geqslant 0,\\ \infty & \text{if } \langle L(\lambda)x, x'\rangle < 0 \text{ for one (and then all) } \lambda > 0.\end{cases}$$

PROPOSITION 4.2. *Let* $\mathscr{X}$ *be an ordered Banach space with normal cone* $\mathscr{X}_+$, *let L be the monic operator polynomial given in* (1) *with nonnegative* *coefficients* A_j $(j = 0, \dots, l-1)$ *and let* $x \in \mathscr{X}_+$, $x \neq 0$. *The following six numbers are equal nonnegative numbers or all are* ∞;

$$\begin{aligned}
&\inf\{\lambda > 0 : L(\lambda)x \geqslant 0\},\\
&\inf\{\lambda > 0 : S(\lambda)x \leqslant \lambda x\},\\
&\inf\{\lambda > 0 : \tilde{r}_{S(\lambda)}(x) \leqslant \lambda\},\\
&\inf\{\max\{\lambda, \tilde{r}_{S(\lambda)}(x)\} : \lambda > 0\},\\
&\inf\{\tilde{r}_C(\mathrm{col}[x, \lambda x, \dots, \lambda^{l-1}x]) : \lambda > 0\},\\
&\sup\{\lambda_{L,x,x'} : x' \in \mathscr{X}_+'\}.
\end{aligned}$$

We omit the proof of this proposition, because it follows the lines of the proof of Proposition 4.1.

If the assumptions of the last two propositions are fulfilled, we call the numbers defined in Propositions 4.1 and 4.2 the lower and upper Collatz-Wielandt number of x with respect to L and denote them by $\underset{\sim}{r}_L(x)$ and $\tilde{r}_L(x)$, respectively. If L is linear in λ, i.e. $L(\lambda) = \lambda I - A$, then it follows immediately that the Collatz-Wielandt numbers of x with respect to L coincide with the corresponding Collatz-Wielandt numbers of x with respect to A. The fundamental relation between the Collatz-Wielandt numbers and the local spectral radius known for nonnegative operators is also true in our case of the monic operator polynomials. We have

THEOREM 4.3. *Let* $\mathscr{X}$ *be an ordered Banach space with normal cone* $\mathscr{X}_+$, *and let L be the monic operator polynomial given in* (1) *with nonnegative coefficients* A_j $(j = 0, \dots, l-1)$. *Then*

$$\underset{\sim}{r}_L(x) \leqslant r_L(x) \leqslant \tilde{r}_L(x) \tag{11}$$

for all $x \in \mathscr{X}_+$, $x \neq 0$.

Proof. For nonnegative operators A we have $\underset{\sim}{r}_A(x) \leqslant r_A(x) \leqslant \tilde{r}_A(x)$ for all $x \in \mathscr{X}_+$, $x \neq 0$; see [5, Proposition 2.3]. The assertion follows from Corollary 3.4(II) and the third characterization of $\underset{\sim}{r}_L(x)$ and $\tilde{r}_L(x)$ in Proposition 4.1 and 4.2, respectively. ■

If we are in the situation that $r_L(x) = r(L)$, see Theorem 3.6, we obtain in Theorem 4.3 a generalization of the classical estimate of the spectral radius of a nonnegative operator by the lower and upper Collatz-Wielandt numbers. Therefore Theorem 4.3 contains the results of [10, Sätze 6, 7] and of [14, Theorems 3.6, 3.7].

If L is linear in λ, i.e. $L(\lambda) = \lambda - A$, and A is a nonnegative operator in $\mathscr{X}$, we have for $x \in \mathscr{X}_+$, $x \neq 0$.

$$\underset{\sim}{r}_A(A^n x) \leqslant \underset{\sim}{r}_A(A^{n+1}x) \leqslant r_A(A^{n+1}x) = r_A(A^n x) = \tilde{r}_A(A^{n+1}x) \leqslant \tilde{r}_A(A^n x) \tag{12}$$

for all $n = 0, 1, \dots$; see [5]. The convergence of the sequences $\{\underset{\sim}{r}_A(A^n x)\}$ and $\{\tilde{r}_A(A^n x)\}$ to $r_A(x)$ has been discussed by several authors, e.g. [9], [17], [15], and [5].

In the last part of this paper we want to give an analogue of the inequality above for arbitrary monic operator polynomials (1) with nonnegative coefficients A_j.

The inequality (9) and the normality of $\mathscr{X}_+$ imply for all nonnegative operators A acting in $\mathscr{X}$, all $x \in \mathscr{X}_+$, and all positive λ and μ that $r_A(S(\lambda)x) = r_A(S(\mu)x)$. If we take $A = S(\lambda)$, we obtain $r_{S(\lambda)}(S(\mu)x) = r_{S(\lambda)}(S(\lambda)x) = r_{S(\lambda)}(x)$ for all $x \in \mathscr{X}_+$ and all positive λ and μ. Therefore

$$r_L(x) = r_L(S(\mu)x) \qquad \text{for all } x \in \mathscr{X}_+ \text{ and all } \mu > 0 \tag{13}$$

is a direct consequence of Corollary 3.4.

Now let $0 < \lambda \leqslant \underset{\sim}{r}_L(x)$, and define $v = S(\lambda)x \in \mathscr{X}_+$. Applying Proposition 4.1 twice, we see that $\lambda \leqslant \underset{\sim}{r}_L(v)$. If μ is such that $\underset{\sim}{r}_L(x) \leqslant \mu < \infty$, we define $w = S(\mu)x \in \mathscr{X}_+$. From Proposition 4.2 we obtain $\tilde{r}_L(w) \leqslant \mu$. From (11) and (13) we obtain

$$\lambda \leqslant \underset{\sim}{r}_L(v) \leqslant r_L(x) \leqslant \tilde{r}_L(w) \leqslant \mu.$$

This inequality shows that for $x \in \mathscr{X}_+$ with $0 < \underset{\sim}{r}_L(x) \leqslant \tilde{r}_L(x) < \infty$ we can define iteratively sequences $\{\lambda_n\}$, $\{\mu_n\}$, $\{v_n\}$, and $\{w_n\}$ by

$$0 < \lambda_1 \leqslant \underset{\sim}{r}_L(x), \qquad v_1 = S(\lambda_1)x$$

$$\lambda_n \leqslant \lambda_{n+1} \leqslant \underset{\sim}{r}_L(v_n), \qquad v_{n+1} = S(\lambda_{n+1})v_n \qquad \text{for} \quad n = 1, 2, \dots,$$

and

$$\tilde{r}_L(x) \leqslant \mu_1, \qquad w_1 = S(\mu_1)x,$$

$$\tilde{r}_L(w_n) \leqslant \mu_{n+1} \leqslant \mu_n, \qquad w_{n+1} = S(\mu_{n+1})w_n \qquad \text{for } n = 1, 2, \dots .$$

From the considerations above it follows immediately that for these sequences

$$\lambda_n \leqslant \lambda_{n+1} \leqslant \underset{\sim}{r}_L(v_n) \leqslant r_L(x) \leqslant \tilde{r}_L(w_n) \leqslant \mu_{n+1} \leqslant \mu_n \tag{14}$$

is true for $n = 1, 2, \dots$. If we take in particular $\lambda_{n+1} = \underset{\sim}{r}_L(v_n)$ and $\mu_{n+1} = \tilde{r}_L(w_n)$ for $n = 1, 2, \dots$ ($v_0 = w_0 = x$), then the sequences $\{\underset{\sim}{r}_L(v_n)\}$ and $\{\tilde{r}_L(w_n)\}$ are nondecreasing and nonincreasing, respectively. In this case (14) is the analogue of (12) for operator polynomials.

The question whether the sequences $\{v_n\}$ and $\{w_n\}$ converge, and the sequences $\{\underset{\sim}{r}_L(v_n)\}$ and $\{\tilde{r}_L(w_n)\}$ converge to $r_L(x)$ were studied in [11] for more general functions L than polynomials, but under very strong conditions on $L(\lambda)$. It is possible to generalize the results of [9], [17], [15], and [5] for the case $L(\lambda) = \lambda I - A$ to the general case of a monic operator polynomial L with arbitrary degree l as defined in (1) with nonnegative coefficients A_j.

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